

THE AUTOMORPHISMS GROUP OF $\overline{M}_{0,n}$

ANDREA BRUNO AND MASSIMILIANO MELLA

INTRODUCTION

The moduli space $M_{g,n}$ of smooth n -pointed curves of genus g , and its projective closure, the Deligne-Mumford compactification $\overline{M}_{g,n}$, is a classical object of study that reflects many of the properties of families of pointed curves. As a matter of fact, the study of its biregular geometry is of interest in itself and has become a central theme in various areas of mathematics.

Already for small n , the moduli spaces $\overline{M}_{0,n}$ are quite intricate objects deeply rooted in classical algebraic geometry. Under this perspective, Kapranov showed in [Ka] that $\overline{M}_{0,n}$ is identified with the closure of the subscheme of the Hilbert scheme parametrizing rational normal curves passing through n points in linearly general position in \mathbb{P}^{n-2} . Via this identification, given $n-1$ points in linearly general position in \mathbb{P}^{n-3} , $\overline{M}_{0,n}$ is isomorphic to an iterated blow-up of \mathbb{P}^{n-3} at the strict transforms of all the linear spaces spanned by subsets of the points in order of increasing dimension. In a natural way, then, base point free linear systems on $\overline{M}_{0,n}$ are identified with linear systems on \mathbb{P}^{n-3} whose base locus is quite special and supported on so-called vital spaces, i.e. spans of subsets of the given points. Another feature of this picture is that all these vital spaces correspond to divisors in $\overline{M}_{0,n}$ which have a modular interpretation as products of $\overline{M}_{0,r}$ for $r < n$. In this interpretation, the modular forgetful maps $\phi_I : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-|I|}$, which forget points indexed by $I \subset \{1, \dots, n\}$, correspond, up to standard Cremona transformations, to linear projections from vital spaces. The aim of this paper is to study automorphisms of $\overline{M}_{0,n}$ with the aid of Kapranov's beautiful description.

It is expected that the only possible biregular automorphisms of $\overline{M}_{0,n}$ are the one associated to a permutation of the markings. Any such morphism has to permute the forgetful maps onto $\overline{M}_{0,n-1}$ as well. This induces, on \mathbb{P}^{n-3} , special birational maps that switch lines through $n-1$ points in general position. On the other hand if we were able to prove that any automorphism has to permute forgetful maps this should lead to a proof that every automorphism is a permutation. Our main tool to classify $\text{Aut}(\overline{M}_{0,n})$ is therefore the following Theorem.

Theorem 1. *Let $f : \overline{M}_{0,n} \rightarrow \overline{M}_{0,r_1} \times \dots \times \overline{M}_{0,r_h}$ be a dominant morphism with connected fibers. Then f is a forgetful map.*

Date: May 2010.

1991 Mathematics Subject Classification. Primary 14H10 ; Secondary 14D22, 14D06.

Key words and phrases. Moduli space of curves, pointed rational curves, fiber type morphism, automorphism.

Partially supported by Progetto PRIN 2008 "Geometria sulle varietà algebriche" MUR.

The above Theorem is an easy extension of the same statement with one factor only and the latter is obtained via an inductive argument starting from the case of a morphism with connected fibers onto \mathbb{P}^1 .

Theorem 2. *Any dominant morphism with connected fibers $f : \overline{M}_{0,n} \rightarrow \overline{M}_{0,4} \cong \mathbb{P}^1$ is a forgetful map.*

The idea of proof is as follows. Any morphism of this type produces a pencil of hypersurfaces on \mathbb{P}^{n-3} . The base locus of this pencil has severe geometric restrictions coming from Kapranov's construction. These are enough to prove that up to a standard Cremona transformation any such pencil is a pencil of hyperplanes.

As already observed this, together with some computation on certain birational endomorphisms of \mathbb{P}^{n-3} , is enough to describe the automorphisms group of $\overline{M}_{0,n}$.

Theorem 3. *Assume that $n \geq 5$, then $\text{Aut}(\overline{M}_{0,n}) = S_n$, the symmetric group on n elements.*

This result has a natural counterpart in the Teichmüller-theoretic literature on the automorphisms of moduli spaces $M_{g,n}$ developed in a series of papers by Royden, Earle–Kra, and others, [Ro], [EK], [Ko], but we do not see a straightforward way to go from one to the other.

In this paper we study “modular” fiber type morphisms on $\overline{M}_{0,n}$ via the study of linear systems on \mathbb{P}^{n-3} and applying whenever possible classical projective techniques. This program has been recently pursued also by Bolognesi, [Bo], in his description of birational models of $\overline{M}_{0,n}$. In a forthcoming paper, [BM], we plan to study fiber type morphisms of $\overline{M}_{0,n}$ onto either low dimensional varieties or with low n or with linear general fiber.

The second named author would like to thank Gavril Farkas for rising his attention on Kapranov's paper, [Ka], and the possibility to use projective techniques in the study of $\overline{M}_{0,n}$. It is a pleasure to thank Gavril Farkas, Angela Gibney, and Daniel Krashen, for their enthusiasm in a preliminary version of these papers and for their suggestions to clarify the contents, many thanks are also due to Brendan Hassett for Teichmüller bibliography and James M^cKernan and Jenia Tevelev for the nice alternative proof of Theorem 3.7.

1. PRELIMINARIES

We work over the field of complex numbers. An n -pointed curve of arithmetic genus 0 is the datum $(C; q_1, \dots, q_n)$ of a tree of smooth rational curves and n ordered points on the nonsingular locus of C such that each component of C has at least three points which are either marked or singular points of C . If $n \geq 3$, $\overline{M}_{0,n}$ is the smooth $(n-3)$ -dimensional scheme constructed by Deligne-Mumford, which is the fine moduli scheme of isomorphism classes $[(C; q_1, \dots, q_n)]$ of stable n -pointed curves of arithmetic genus 0.

For any $i \in \{1, \dots, n\}$ the forgetful map

$$\phi_i : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-1}$$

is the surjective morphism which associates to the isomorphism class $[(C; q_1, \dots, q_n)]$ of a stable n -pointed rational curve $(C; q_1, \dots, q_n)$ the isomorphism class of the $(n-1)$ -pointed stable rational curve obtained by forgetting q_i and, if any, contracting to a point a component of C containing only q_i , one node of C and another

marked point, say q_j . The locus of such curves forms a divisor, that we will denote by $E_{i,j}$. The morphism ϕ_i also plays the role of the universal curve morphism, so that its fibers are all rational curves transverse to $n-1$ divisors $E_{i,j}$. Divisors $E_{i,j}$ are the images of $n-1$ sections $s_{i,j} : \overline{M}_{0,n-1} \rightarrow \overline{M}_{0,n}$ of ϕ_i . The section $s_{i,j}$ associates to $[(C; q_1, \dots, q_i^\vee, \dots, q_n)]$ the isomorphism class of the n -pointed stable rational curve obtained by adding at q_j a smooth rational curve with marking of two points, labelled by q_i and q_j . Analogously, for every $I \subset \{1, \dots, n\}$, we have well defined forgetful maps $\phi_I : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-|I|}$. From our point of view the important part of a forgetful map is the set of forgotten index, more than the actual marking of the remaining. For this we slightly abuse the language and introduce the following definition.

Definition 1.1. A forgetful map is the composition of $\phi_I : \overline{M}_{0,n} \rightarrow \overline{M}_{0,r}$ with an automorphism $g \in \text{Aut}(\overline{M}_{0,r})$ that permutes the markings.

In order to avoid trivial cases we will always tacitly consider ϕ_I only if $n - |I| \geq 4$.

Besides the canonical class $K_{\overline{M}_{0,n}}$, on $\overline{M}_{0,n}$ are defined line bundles Ψ_i for each $i \in \{1, \dots, n\}$ as follows: the fiber of Ψ_i at a point $[(C; q_1, \dots, q_n)]$ is the tangent line T_{C,p_i} . Kapranov, in [Ka] proves the following:

Theorem 1.2. *Let $p_1, \dots, p_n \in \mathbb{P}^{n-2}$ be points in linear general position. Let H_n be the Hilbert scheme of rational curves of degree $n-2$ in \mathbb{P}^{n-2} . $\overline{M}_{0,n}$ is isomorphic to the subscheme $H \subset H_n$ parametrizing curves containing p_1, \dots, p_n . For each $i \in \{1, \dots, n\}$ the line bundle Ψ_i is big and globally generated and it induces a morphism $f_i : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ which is an iterated blow-up of the projections from p_i of the given points and of all strict transforms of the linear spaces they generate, in order of increasing dimension.*

We will use the following notations:

Definition 1.3. A Kapranov set $\mathcal{K} \subset \mathbb{P}^{n-3}$ is an ordered set of $(n-1)$ points in linear general position, labelled by a subset of $\{1, \dots, n\}$. For any $J \subset \mathcal{K}$, the linear span of points in J is said a vital linear subspace of \mathbb{P}^{n-3} . A vital cycle is any union of vital linear subspaces.

To any Kapranov set, labelled by $\{1, \dots, i-1, i+1, \dots, n\}$, is uniquely associated a Kapranov map, $f_i : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$, with $\Psi_i = f_i^* \mathcal{O}_{\mathbb{P}^{n-3}}(1)$, and to a Kapranov map is uniquely associated a Kapranov set up to projectivity.

Definition 1.4. Given a subset $I = \{i, i_1, \dots, i_s\} \subset \{1, \dots, n\}$ and the Kapranov map $f_i : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$, let $I^* = \{1, \dots, n\} \setminus I$. Then we indicate with

$$H_{I^*}^{i\vee} := V_{I \setminus \{i\}}^i := V_{i_1, \dots, i_s}^i := \langle p_{i_1}, \dots, p_{i_s} \rangle \subset \mathbb{P}^{n-3}$$

the vital linear subspace generated by the p_{i_j} 's and with

$$E_I := E_{i, i_1, \dots, i_s} := f_i^{-1}(V_I^i)$$

the divisor associated on $\overline{M}_{0,n}$.

Notice that $H_{ij}^{h\vee}$ is the hyperplane missing the points p_i and p_j and the set

$$\mathcal{K}' = \mathcal{K} \setminus \{p_i, p_j\} \cup (H_{ij}^{h\vee} \cap \langle p_i, p_j \rangle)$$

is a Kapranov set in $H_{ij}^{h\vee}$.

In particular for any $i \in \{1, \dots, n\}$ and Kapranov set $\mathcal{K} = \{p_1, \dots, p_i^\vee, \dots, p_n\}$ divisors $E_{i,j} = f_i^{-1}(p_j)$ are defined and such notation is compatible with the one adopted for the sections $E_{i,j}$ of ϕ_i . More generally, for any $i \in I \subset \{1, \dots, n\}$ the divisor E_I has the following property: its general point corresponds to the isomorphism class of a rational curve with two components, one with $|I| + 1$ marked points, the other with $|I^*| + 1$ marked points, glued together at the points not marked by elements of $\{1, \dots, n\}$. It follows from this picture that $E_I = E_{I^*}$ and that E_I is abstractly isomorphic to $\overline{M}_{0,|I|+1} \times \overline{M}_{0,|I^*|+1}$. The divisors E_I parametrise singular rational curves, and they are usually called boundary divisors. A further property of E_I is that for each choice of $i \in I, j \in I^*$, E_I is a section of the forgetful morphism:

$$\phi_{I \setminus \{i\}} \times \phi_{I^* \setminus \{j\}} : \overline{M}_{0,n} \rightarrow \overline{M}_{0,|I^*|+1} \times \overline{M}_{0,|I|+1}.$$

This morphism is surjective and all fibers are rational curves. With our notations $f_i(E_I)$ is a vital linear space of dimension $|I| - 2$ if $i \in I$ and a vital linear space of dimension $|I^*| - 2$ if $i \notin I$.

Definition 1.5. A dominant morphism $f : X \rightarrow Y$ is called a fiber type morphism if the dimension of the general fiber is positive, i.e. $\dim X > \dim Y$.

We are interested in describing linear systems on \mathbb{P}^{n-3} that are associated to fiber type morphisms on $\overline{M}_{0,n}$. For this purpose we introduce some definitions.

Definition 1.6. A linear system on a smooth projective variety X is uniquely determined by a pair (L, V) , where $L \in \text{Pic}(X)$ is a line bundle and $V \subseteq H^0(X, L)$ is a vector space. If no confusion is likely to arise we will forget about V and let $\mathcal{L} = (L, V)$. Let $g : Y \rightarrow X$ be a birational morphism between smooth varieties. Let $A \in \mathcal{L}$ be a general element and $A_Y = g_*^{-1}A$ the strict transform. Then $g^*L = A_Y + \Delta$ for some effective g -exceptional divisor Δ . The strict transform of $\mathcal{L} = (L, V)$ via g is

$$g_*^{-1}\mathcal{L} := (g^*L - \Delta, V_Y)$$

where V_Y is the vector space spanned by the strict transform of elements in V .

Definition 1.7. Let $\mathcal{K} \subset \mathbb{P}^{n-3}$ be a Kapranov set and $f_i : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ the associated map. An $M_{\mathcal{K}}$ -linear system on \mathbb{P}^{n-3} is a linear system $\mathcal{L} \subseteq |\mathcal{O}_{\mathbb{P}^{n-3}}(d)|$, for some d , such that $f_{i*}^{-1}\mathcal{L}$ is a base point free linear system.

Let \mathcal{L} be an $M_{\mathcal{K}}$ -linear system, and fix $f_1 : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ a Kapranov map. To better understand the properties of $M_{\mathcal{K}}$ -linear systems let us look closer at f_1 . Let $\epsilon : Y \rightarrow \mathbb{P}^{n-3}$ be the blow up of $p_2 \in \mathcal{K}$ with exceptional divisor E . Then Kapranov's map f_1 , can be factored as follows

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{g} & Y \\ \downarrow f_1 & \nearrow \epsilon & \\ \mathbb{P}^{n-3} & & \end{array}$$

with a birational morphism $g : \overline{M}_{0,n} \rightarrow Y$. The map g is obtained by blowing up, in the prescribed order, the strict transform of every vital cycle of codimension at least 2 in Y . In particular it is an isomorphism on every codimension 1 point of $E \subset Y$. With this observation we are able to weakly control the base locus of $\mathcal{L}_Y := \epsilon_*^{-1}\mathcal{L}$, the strict transform linear system.

Lemma 1.8. *Let \mathcal{L} be an $M_{\mathcal{K}}$ -linear system without fixed components, associated to the Kapranov map $f_i : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$. Let $\epsilon : Y \rightarrow \mathbb{P}^{n-3}$ be the blow up of the Kapranov point $p_j \in \mathcal{K}$ with exceptional divisor E . Let \mathcal{L}_Y be the strict transform. Then the linear system $\mathcal{L}_{Y|E}$ has not fixed components.*

Proof. Let

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{g} & Y \\ \downarrow f_i & \nearrow \epsilon & \\ \mathbb{P}^{n-3} & & \end{array}$$

be the commutative diagram as above, with exceptional divisor $E \subset Y$. We noticed that g is an isomorphism on every codimension 1 point of E . By hypothesis \mathcal{L} and hence \mathcal{L}_Y have not fixed components. By construction g is a resolution of $\text{Bs } \mathcal{L}_Y$. This yields that

$$\text{cod}_E(\text{Bs } \mathcal{L}_Y \cap E) \geq 2,$$

and $\mathcal{L}_{Y|E}$ has not fixed components. \square

A further property inherited from Kapranov's construction is the following.

Remark 1.9. *Let $H_{ij}^{h\vee}$ be the hyperplane missing the points p_i and p_j and*

$$\mathcal{K}' = \mathcal{K} \setminus \{p_i, p_j\} \cup (H_{ij}^{h\vee} \cap \langle p_i, p_j \rangle)$$

the associated Kapranov set. Then $\mathcal{L}_{|H_{ij}^{h\vee}}$ is an $M_{\mathcal{K}'}$ linear system.

Basic examples of fiber type morphisms are forgetful maps. Consider any set $I \subset \{1, \dots, n\}$ and the associated forgetful map ϕ_I . If $j \notin I$ a typical diagram we will consider is

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{\phi_I} & \overline{M}_{0,n-|I|} \\ \downarrow f_j & & \downarrow f_h \\ \mathbb{P}^{n-3} & \xrightarrow{\pi_I} & \mathbb{P}^{n-|I|} \end{array}$$

where the f_j and f_h are Kapranov maps and π_I is the projection from V_I^j . In this case an $M_{\mathcal{K}}$ -linear system associated to Φ_I is given by $|\mathcal{O}(1) \otimes \mathcal{I}_{V_I^j}|$ and if F_I is any fiber of ϕ_I , $f_j(F_I)$ is a linear space of codimension $|I|$.

It is important, for what follows, to understand explicitly the rational map π_I when $j \in I$. To do this we use Cremona transformations.

2. CREMONA TRANSFORMATIONS AND $M_{\mathcal{K}}$ -LINEAR SYSTEMS

Definition 2.1. Let \mathcal{K} be a Kapranov set with Kapranov map $f_i : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$. Then let

$$\omega_j^{\mathcal{K}} : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-3}$$

be the standard Cremona transformation centered on $\mathcal{K} \setminus \{p_j\}$. Via Kapranov's construction we can associate a Kapranov set labelled by $\{1, \dots, n\} \setminus \{j\}$ to the rhs \mathbb{P}^{n-3} and in this notation Kapranov, [Ka, Proposition 2.14] proved that

$$\omega_j^{\mathcal{K}} = f_j \circ f_i^{-1}$$

as birational maps. By a slight abuse of notation we can define $\omega_j^{\mathcal{K}}(V_I^i) := f_j(E_{I,i})$, even if ω_j is not defined on the general point of V_I^i .

Remark 2.2. Let ω_h^K be the standard Cremona transformation centered on $\mathcal{K} \setminus \{p_h\}$, and \mathcal{K}' the Kapranov set associated to the hyperplane $H_{jk}^{i\vee}$. Then for $h \neq j, k$ we have $\omega_h^K|_{H_{jk}^{i\vee}} = \omega_h^{K'}$. This extends to arbitrary vital linear spaces. It follows from the definitions that $\omega_h^K(V_I^i) = V_{I \setminus \{h\}, i}^h$ if $h \in I$ and $\omega_h^K(V_I^i) = V_{(I \cup \{i\})^* \setminus \{h\}}^h$ if $h \in I^*$.

Let us start with the special case of forgetful maps onto $\overline{M}_{0,4} \cong \mathbb{P}^1$. Let

$$\phi_I : \overline{M}_{0,n} \rightarrow \mathbb{P}^1 \cong \overline{M}_{0,4}$$

be a forgetful map and $\mathcal{M} = \phi_I^* \mathcal{O}(1)$. Choose a Kapranov map $f_i : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ with $\mathcal{L} := f_{i*} \mathcal{M} \subset |\mathcal{O}(1)|$. As already noticed this is equivalent to choose $i \notin I$. Then $\text{Bs } \mathcal{L} = P$ is a codimension 2 linear space and we may assume, after reordering the indexes, that

$$\mathcal{K} \setminus (\text{Bs } \mathcal{L} \cap \mathcal{K}) = \{p_1, p_2, p_3\} \text{ and } i = 4.$$

To understand what is the linear system $\mathcal{L}_5 := f_{5*} \mathcal{M}$ we use Kapranov description of the map ω_5^K , see Definition 2.1.

This is well known but we decided to write it down for readers less familiar with the classical subject of Cremona Transformations. The map is the standard Cremona transformation centered on $\{p_1, p_2, p_3, p_6, \dots, p_n\}$. Let

$$\omega_5^K : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-3} =: \mathbb{P}$$

be the map given by the linear system

$$|\mathcal{O}_{\mathbb{P}^{n-3}}(n-3) \otimes (\otimes_{i \in \{1,2,3,6,\dots,n\}} \mathcal{I}_{p_i}^{n-4})|.$$

Let

$$\begin{array}{ccc} & Z & \\ p \swarrow & & \searrow q \\ \mathbb{P}^{n-3} & \xrightarrow{\omega_5^K} & \mathbb{P} \end{array}$$

be the usual resolution obtained by blowing up, in dimension increasing order, all linear spaces spanned by points in $\{p_1, p_2, p_3, p_6, \dots, p_n\}$. Let $l \subset \mathbb{P}$ be a general line then q^{-1} is well defined on l and $p(q^{-1}(l))$ is a rational normal curves passing through $\{p_1, p_2, p_3, p_6, \dots, p_n\}$. Let $E_i \subset Z$ be the exceptional divisor corresponding to the blow up of the points p_i . Then we have $q^{-1}(l) \cdot E_i = 1$, for $i \in \{1, 2, 3, 6, \dots, n\}$. While $q^{-1}(l) \cdot F$ vanishes for any other p -exceptional divisor. This description allows us to easily compute the degree of \mathcal{L}_5

$$\deg \mathcal{L}_5 = \deg \omega_5^K(\mathcal{L}) = (n-3) - \sum_{h \neq 5} \text{mult}_{p_h} \mathcal{L} = 2$$

This yields $\omega_5^K(\mathcal{L}) \subset |\mathcal{O}(2)|$.

To complete the analysis we have to understand the base locus of this system of quadrics. Let \mathcal{K}_5 be the Kapranov set labelled by $\{1, 2, 3, 4, 6, \dots, n\}$.

In our convention, see Definition 2.1, for $\{i, j\} \subset \{1, 2, 3\}$ we have

$$\omega_5^K(V_{i,j}^4) = V_{\{i,j,4\}^* \setminus \{5\}}^5$$

That is this line is sent to a codimension two linear space. Let $P_h = \omega_5^K(V_{i,j}^4)$, for $\{i, j, h\} = \{1, 2, 3\}$. The general element in \mathcal{L} intersects a general point of $V_{i,j}^4$ and therefore its transform via ω_5^K has to contain P_h . The hypothesis $5 \in I$ tell us that

the map ω_5^K is well defined on the general point of $P = V_I^4$. Let $P_4 = \omega_5^K(P)$, and $S := \omega_5^K(\langle p_1, p_2, p_3 \rangle)$. Then P_4 has to be contained in $\text{Bs } \mathcal{L}_5$ and

$$S = \cap_{i=1}^4 P_i.$$

In conclusion we have:

- . $\text{Bs } \mathcal{L}_4 = \cup_{i=1}^4 P_i \supset \mathcal{K}_5$
- . $\text{Sing}(\mathcal{L}_5) = S$.

Hence the linear system \mathcal{L}_5 is a pencil of quadrics with four codimension two linear spaces in the base locus. That is the cone, in \mathbb{P}^{n-3} , over a pencil of conics through 4 general points and vertex $V_{I \setminus \{5\}}^5$.

To study fiber type morphisms from $\overline{M}_{0,n}$ it is important to control the base locus of M_K -linear systems. The easiest base loci are those of forgetful maps $\phi_I : \overline{M}_{0,n} \rightarrow \overline{M}_{0,r}$. For this we introduce the following definitions.

Definition 2.3. Let $\pi_i : \mathbb{P}^{r-2} \rightarrow \mathbb{P}^{r-3}$ be the projection from a Kapranov point p_i , and $\mathcal{L} = |\mathcal{O}_{\mathbb{P}^{r-2}}(1) \otimes \mathcal{I}_{p_i}|$. Define

$$\mathcal{C}_{r-3}^i := \omega_i^K(\mathcal{L}) \subset |\mathcal{O}_{\mathbb{P}^{r-2}}(r-2)|,$$

to be the transform of hyperplanes through the point p_i . We say that an M_K -linear system \mathcal{M} on \mathbb{P}^{n-3} , has base locus of type Φ_r if $\text{Bs } \mathcal{M}$ is either a codimension $r-2$ linear space or the cone over $\text{Bs } \mathcal{C}_{r-3}^i$ with vertex a linear space of codimension $r-1$. Equivalently \mathcal{M} has base locus of type Φ_r if it is an M_K -linear system with linear base locus of codimension $r-2$ up to standard Cremona transformations.

In this notation \mathcal{C}_1^i is a pencil of plane conics through 4 fixed points. The above construction shows that to a forgetful map $\phi_I : \overline{M}_{0,n} \rightarrow \overline{M}_{0,4}$ are associated M_K -linear systems with base locus of type Φ_1 . This is actually the main motivation of our definition. We will use, and improve this observations first in Proposition 2.4 and further in Lemma 3.5. The main point in our construction is that the base locus of M_K -linear system is enough to characterise linear systems inherited by forgetful maps.

The special case of forgetful maps onto \mathbb{P}^1 is the one we use in this paper. Nonetheless we would like to stress that a similar behaviour applies to an arbitrary forgetful map onto $\overline{M}_{0,r}$, for $r < n$.

Proposition 2.4. Let $\phi_I : \overline{M}_{0,n} \rightarrow \overline{M}_{0,r}$ be a forgetful morphism. Assume that $1 \in I$ and let

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{\phi_I} & \overline{M}_{0,r} \\ f_1 \downarrow & & \downarrow f_i \\ \mathbb{P}^{n-3} & \xrightarrow{\pi_I} & \mathbb{P}^{r-3} \end{array}$$

be the usual diagram. Then π_I is given by a sublinear system $\mathcal{L}_1 \subset |\mathcal{O}(r-2)|$, the general fiber of π_I is a cone, with vertex $V_{I \setminus \{1\}}^1 \subset \mathbb{P}^{n-3}$, over a rational normal curve of degree $n-2-|I|$, and \mathcal{L}_1 has Base locus of type Φ_r .

Proof. The morphism Φ_I can be factored as follows

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{\phi_{I \setminus \{2\}}} & \overline{M}_{0,r+1} \\ \downarrow \phi_I & \searrow \phi_2 & \\ \overline{M}_{0,r} & & \end{array}$$

Hence we have the induced diagram

$$\begin{array}{ccccc} \overline{M}_{0,n} & \xrightarrow{\phi_{I \setminus \{2\}}} & \overline{M}_{0,r+1} & & \\ \downarrow \phi_I & \searrow \phi_2 & \searrow f_2 & & \\ \overline{M}_{0,r} & & \mathbb{P}^{n-3} & \xrightarrow{\pi_{I \setminus \{2\}}} & \mathbb{P}^{r-2} \\ & \searrow f_r & \downarrow \pi_I & \swarrow \varphi & \\ & & \mathbb{P}^{r-3} & & \end{array}$$

where $\pi_{I \setminus \{2\}}$ is a linear projection and φ is the map induced by \mathcal{C}_r^2 . The claim then follows. \square

3. BASE POINT FREE PENCILS ON $\overline{M}_{0,n}$

A base point free pencil \mathcal{L} on $\overline{M}_{0,n}$ is the datum of a couple (L, V) on $\overline{M}_{0,n}$, where L is a line bundle on $\overline{M}_{0,n}$ and $V \subset H^0(\overline{M}_{0,n}, L)$ is a two-dimensional subspace. The natural map $V \otimes \mathcal{O}_{\overline{M}_{0,n}} \rightarrow L$ is surjective and this datum is equivalent to a surjective morphism $f : \overline{M}_{0,n} \rightarrow \mathbb{P}^1$ such that $L = f^*(\mathcal{O}(1))$.

Fix a Kapranov map $f_i : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ and let $\mathcal{L}_i := f_* \mathcal{L}$. Then the linear system \mathcal{L}_i is an M_K -linear system inducing a birational map $\pi^i : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^1$. Moreover $\mathcal{L} = f_{i*}^{-1} \mathcal{L}_i$ and f is a, not necessarily minimal, resolution of the indeterminacy of the map π^i . To the map f we therefore associate a diagram

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{f} & \mathbb{P}^1 \\ f_i \downarrow & & \downarrow \cong \\ \mathbb{P}^{n-3} & \xrightarrow[\pi^i]{\dots\dots\dots} & \mathbb{P}^1 \end{array}$$

where $\pi^i := f \circ f_i^{-1}$. The rational map π^i is uniquely associated to a pencil $\mathcal{L}_i \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(d_i)|$ free of fixed divisors on \mathbb{P}^{n-3} . We have $\mathcal{L}_i = (\mathcal{O}_{\mathbb{P}^{n-3}}(d_i), W_i)$ where $W_i \subset H^0(\mathbb{P}^{n-3}, \mathcal{O}(d_i))$ has dimension two, and any element of V is the strict transform of an element of W_i , i.e. the strict transform map $f_{i*}^{-1} : W_i \rightarrow V$ is an isomorphism and the support of the cokernel of the evaluation map

$$ev_i : W_i \otimes \mathcal{O}_{\mathbb{P}^{n-3}} \rightarrow \mathcal{L}_i$$

on \mathbb{P}^{n-3} does not have divisorial components. In particular the support of the cokernel of ev_i is the base locus $\text{Bs } \mathcal{L}_i$ of \mathcal{L}_i .

The most important example of dominant maps $f : \overline{M}_{0,n} \rightarrow \mathbb{P}^1$ is given by the forgetful maps already described in Section 2. The goal of this section will be to prove that in fact any surjective map with connected fibers $f : \overline{M}_{0,n} \rightarrow \mathbb{P}^1$ is in fact

a forgetful map. The criterion we are going to use in order to understand whether a morphism $f : \overline{M}_{0,n} \rightarrow \mathbb{P}^1$ is a forgetful map is the following:

Proposition 3.1. *Let $f : \overline{M}_{0,n} \rightarrow X$ be a surjective morphism. Let $A \in \text{Pic}(X)$ be a base point free linear system and $\mathcal{L}_i = f_{i*}(f^*(A))$. Assume that for some j $\text{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i$. Then f factors through the forgetful map $\phi_j : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-1}$.*

Let $f : \overline{M}_{0,n} \rightarrow \overline{M}_{0,r}$ be a surjective morphism and $\pi : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{r-3}$ the induced map. Let $\mathcal{L}_i = f_{i}(f^*(g_{j*}^{-1}(\mathcal{O}(1)))$ and assume that $\mathcal{L}_i \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|$. Then f is a forgetful map.*

Proof. Let $\phi_j : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-1}$ be the forgetful morphism. Fibers of ϕ_j are mapped by f_i to lines in \mathbb{P}^{n-3} through the point p_j . If $\text{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i$ the restriction of \mathcal{L}_i to each such line has a trivial moving part. In particular the linear system $f^*(A)$ is base point free and numerically trivial on every fiber of ϕ_j . Moreover $\text{Pic}(\overline{M}_{0,n}/\overline{M}_{0,n-1}) = \text{Num}(\overline{M}_{0,n}/\overline{M}_{0,n-1})$, therefore $f^*(A)$ is ϕ_j -trivial. This shows that fibers of ϕ_j are contracted by f . For any $h \neq i, j$ the map ϕ_j has a section $s_{j,h} : \overline{M}_{0,n-1} \rightarrow E_{j,h} \subset \overline{M}_{0,n}$, described in Section 1 and then a morphism $g := f \circ s_{j,h} : \overline{M}_{0,n-1} \rightarrow X$ is given such that $f = g \circ \phi_j$. Notice that g does not depend on the choice of $h \neq i, j$.

Assume that $\mathcal{L}_i = |\mathcal{O}_{\mathbb{P}^{n-3}}(1) \otimes \mathcal{I}_{V_I^i}|$ for some vital cycle $V_I^i \subset \mathbb{P}^{n-3}$. Then for any vital point $p_k \in V_I^i$ we have $\text{mult}_{p_k} \mathcal{L}_i = \deg \mathcal{L}_i$. Then by the first statement we have that ϕ_k factors f for any $k \in I$. Then there is a map $g : \overline{M}_{0,r} \rightarrow \overline{M}_{0,r}$ such that $f = g \circ \phi_I$. Let $\gamma : \mathbb{P}^{r-3} \dashrightarrow \mathbb{P}^{r-3}$ be the induced map. Then γ is associated to a linear system of hyperplanes and it is therefore a projectivity that eventually permutes the Kapranov set on \mathbb{P}^{r-3} , keep in mind our Definition 1.1. \square

Definition 3.2. Let $\mathcal{L}_i := (L_i, W_i)$ be an M_K -linear system on \mathbb{P}^{n-3} and $A_i \in \mathcal{L}_i$ a general element. Let $H = H_{h,k}^{\vee}$ be a vital hyperplane. We say that the restriction of \mathcal{L}_i to H is dominant if the restriction map $\text{res}_H : W_i \rightarrow H^0(H, L_i|_H)$ is injective. Let $p_j \in \mathcal{K}$ be a Kapranov point. Let $\epsilon_j : Y_j \rightarrow \mathbb{P}^{n-3}$ be the blow-up of p_j with exceptional divisor E_j . Assume that $\epsilon_j^* A_i = A_{iY} + mE_j$. Then

$$\mathcal{L}_{i,Y_j} := (\epsilon_j^* L_i - mE_j, W_i^Y)$$

is the strict transform of \mathcal{L}_i . We say that \mathcal{L}_i is dominant at the first order of p_j if the pullback map $\epsilon_j^* : W_i \rightarrow H^0(E_j, (\epsilon_j^* L_i - mE_j)|_{E_j})$ is injective and $\mathcal{L}_{i,Y_j|E_j}$ is without fixed divisors.

We sketch here the ideas underlying our argument in order to characterise base point free pencils on $\overline{M}_{0,n}$.

We proceed by induction on n and assume that all base point free pencils on $\overline{M}_{0,n-1}$ inducing a surjective map with connected fibers $f : \overline{M}_{0,n-1} \rightarrow \mathbb{P}^1$ are forgetful maps. The well known case of $n = 5$ is the beginning of the induction argument. According to our criterion, Proposition 3.1, for the induction step it is enough to show that there exists a Kapranov map f_i and a Kapranov point p_j such that

$$\text{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i.$$

To produce this point we find a vital hyperplane H such that the restriction of \mathcal{L}_i to H is dominant. Then the hyperplane H has a Kapranov set and it is the image under a Kapranov map of $\overline{M}_{0,n-1}$, see remark 1.9. By induction we may find the

required point for $\mathcal{L}_i|_H$ and then lift it to the linear system \mathcal{L}_i . Here is a list of concerns in applying this idea:

- . How can we find a vital hyperplane H such that the restriction of \mathcal{L}_i to it is dominant ?
- . How to compare $\text{mult}_{p_j} \mathcal{L}_i|_H$ with $\text{mult}_{p_j} \mathcal{L}_i$?
- . What if the restricted morphism has either non connected fibers or fixed components ?

As a matter of fact, even if (L_i, W_i) is free of fixed divisors and if it induces a map with connected fibers, this may not be the case for the restricted linear system on any vital hyperplane $H \subset \mathbb{P}^{n-3}$. Keep in mind that there are only finitely many of those.

The desired hyperplane H is produced in Lemmata 3.3 and 3.4. The basic idea is that the base locus of \mathcal{L}_i cannot be empty because \mathbb{P}^{n-3} does not carry base point free pencils so that there exists some point p_j contained in the base locus of \mathcal{L}_i . Notice that this does not mean that such a point is an isolated component of $\text{Bs } \mathcal{L}_i$. Thanks to Lemma 1.8 we can prove that the M_K -linear system (L_i, W_i) is dominant at the first order of p_j . To apply induction on the exceptional divisor over the point p_j we have to study pencils with possibly non connected fibers. This is done in Lemma 3.5. With this and induction hypothesis we know that the pencil induced on the exceptional divisor has base locus of type Φ_1 . Hence we may apply induction and find a hyperplane H such that \mathcal{L}_i restricted to H is dominant.

Finally, we use Lemma 3.6 in order to show that we can in fact exclude the presence of fixed divisors on the restricted linear systems, so that we can really infer properties of (L_i, W_i) from properties of the restriction to some hyperplane H .

We now prove the above mentioned Lemmata. Let us fix a pencil \mathcal{L}_i , without fixed components, together with the usual diagram

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{f} & \mathbb{P}^1 \\ f_i \downarrow & & \downarrow \cong \\ \mathbb{P}^{n-3} & \xrightarrow[\pi^i]{\dots\dots\dots} & \mathbb{P}^1 \end{array}$$

and notation. Let $p_j \in \mathcal{K} \cap \text{Bs } \mathcal{L}_i$ be a point and $\epsilon_j : Y_j \rightarrow \mathbb{P}^{n-3}$ the blow-up of p_j with exceptional divisor E_j . Let $\mathcal{L}_{i,Y_j} = \epsilon_j^* \mathcal{L}_i - m_j E_j$ be the strict transform of \mathcal{L}_i , for some positive m_j .

Lemma 3.3. *The linear system $\mathcal{L}_i = (L_i, W_i)$ is dominant at the first order of p_j and for any $A_1, A_2 \in \mathcal{L}_i$ we have*

$$\text{mult}_{p_j} A_1 = \text{mult}_{p_j} A_2.$$

Let $H = H_{hk}^{i\vee}$ be a vital hyperplane containing p_j and $A \in \mathcal{L}_i$ a general element. Then we have

$$\text{mult}_{p_j} A = \text{mult}_{p_j} A|_H$$

Proof. In the above notation we know, by Lemma 1.8, that $\mathcal{L}_{i,Y_j|E_j}$ has not fixed components. Hence the image of the pullback map $\epsilon_j^* : W_i \rightarrow H^0(E_j, \mathcal{L}_{i,Y_j|E_j})$ is

not one dimensional. Since $\dim W_i = 2$ we conclude that ϵ_j^* is injective as required. The injectivity of ϵ_j^* forces every element in \mathcal{L}_i to have the same multiplicity at p_j .

Let H_{Y_j} be the strict transform of H on Y_j . Then again by Lemma 1.8 we know that $\text{Bs } \mathcal{L}_{i,Y_j} \not\supset E \cap H_{Y_j}$. Therefore the general element $A \in \mathcal{L}_i$ satisfies $\text{mult}_{p_j} A = \text{mult}_{p_j} A|_H$. \square

Lemma 3.4. *Let $H = V_I^i \subset \mathbb{P}^{n-3}$ be a vital hyperplane such that $p_j \in H$. If $f(E_{i,I})$ is a point and if H_j is the strict transform of H under ϵ_j , then $L_{i,Y_j}|_{E_j}$ is trivial along $H_j \cap E_j$.*

Proof. The morphism f is a resolution of indeterminacies of π^i and f_i factors through ϵ_j . Then we have the result. \square

In Section 2 we proved that every forgetful map onto \mathbb{P}^1 induces an $M_{\mathcal{K}}$ -linear system of degree at most 2 with base locus of type Φ_1 . One cannot expect that all morphisms to \mathbb{P}^1 have bounded degree. On the other hand, under suitable hypothesis, the base locus of $M_{\mathcal{K}}$ -linear systems is unaffected by connectedness of fibers.

Lemma 3.5. *Assume that every dominant morphism $g : \overline{M}_{0,n} \rightarrow \mathbb{P}^1$ with connected fibers is a forgetful map. Then $\text{Bs } \mathcal{L}_i$ is of type Φ_1 . If moreover $n \geq 7$ there are vital points p_j satisfying $\text{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i$.*

Proof. Let $h : \overline{M}_{0,n} \rightarrow C$ be the Stein factorization of f . Let $\nu : \tilde{C} \rightarrow C$ be the normalization. Then there is a unique map $f' : \overline{M}_{0,n} \rightarrow \tilde{C}$ such that $h = \nu \circ f'$. The variety $\overline{M}_{0,n}$ is rational, therefore $\tilde{C} \cong \mathbb{P}^1$ and $|f^* \mathcal{O}(1)| \subset |f'^* \mathcal{O}(\gamma)|$ for some integer γ . By hypothesis f' is a forgetful map therefore we may choose i in such a way that $|f_{i*} f'^* \mathcal{O}(1)| \subset |\mathcal{O}_{P^{n-3}}(1)|$ satisfies

$$\text{Bs } |f_{i*} f'^* \mathcal{O}(1)| = V_I^j,$$

where V_I^j is a codimension two irreducible vital space. Then the elements in the linear system \mathcal{L}_i are union of γ hyperplanes containing V_I^j and

$$\text{Bs } \mathcal{L}_i := \text{Bs } |f_{i*} f'^* \mathcal{O}(1)| = \text{Bs } |f_{i*} f'^* \mathcal{O}(1)| = V_I^j.$$

To conclude it is enough to apply standard Cremona transformations to this configuration as described in Section 2. In particular if $n \geq 7$ all linear systems of type Φ_1 are cones with non empty vertex and therefore there is at least a point p_j with $\text{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i$. \square

We conclude this technical part taking into account the eventual fixed divisors.

Lemma 3.6. *Let $H = H_{h,k}^{i\vee}$ be a vital hyperplane in \mathbb{P}^{n-3} , with $j \neq h, k$. Assume that $\mathcal{L}_i = (L_i, W_i)$ has a dominant restriction to H . Assume that F is the fixed divisor of the restricted system $\mathcal{L}_i|_H$ and that $p_j \notin F$. Then*

$$F = \langle p_l | l \neq h, k, j \rangle = V_{\{i,h,k,j\}}^{i*}.$$

Proof. The fixed divisor $F \subset H$ is in the base locus of (L_i, W_i) . By hypothesis \mathcal{L}_i has not fixed divisors. Then the support of F must be an irreducible component of $\text{Bs } \mathcal{L}_i$. In particular F does not contain p_h, p_k , and therefore cannot intersect the line $\langle p_h, p_k \rangle = V_{h,k}^i$. By hypothesis we have $p_j \notin F$. Hence the only possibility left is $F = \langle p_l | l \neq h, k, j \rangle$. \square

Our inductive argument starts with the classical case of $\overline{M}_{0,5}$. The next step is a special case of [FG] study of the cone of effective curves of $\overline{M}_{0,6}$. From our point of view the $n = 6$ case is a bit more complicate because in this case it is not true that there is always a point with $\text{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i$. The pencil \mathcal{C}_1^i is a pencil of plane conics. Nonetheless we prefer to prove the $n = 6$ case with our techniques to show a slightly more complicate application of Cremona transformations $\omega_h^{\mathcal{K}}$. Suppose $\mathcal{L} = (L, V)$ is a pencil on $\overline{M}_{0,6}$. Choose a Kapranov map f_i and the induced pencil $\mathcal{L}_i = (L_i, W_i)$ on \mathbb{P}^3 . From Lemma 3.3 we find a point $p_j \in \mathcal{K} \cap \text{Bs } \mathcal{L}_i$ such that \mathcal{L}_i dominates p_j at first order. Let E_j be the exceptional divisor of the blow-up of p_j . This is a plane with a natural Kapranov set induced by the lines in \mathbb{P}^3 joining p_j with any other point of $\mathcal{K} \subset \mathbb{P}^3$. Furthermore the strict transform \mathcal{L}_{i,Y_j} is in a natural way an $M_{\mathcal{K}}$ -linear system. We may apply Lemma 3.5 in order to deduce that its base locus is of type Φ_1 . After possibly switching the Kapranov map, see Remark 2.2, we may assume that $\text{Bs } \mathcal{L}_{i,Y_j}$ consists of one point, say p_l , and that fibers of the rational map to \mathbb{P}^1 are sets of lines through p_l . This, together with Lemma 3.4 yields that if $H \subset \mathbb{P}^3$ is a vital hyperplane containing p_j and not containing p_l , the restriction of (L_i, W_i) to H is dominant. We may assume that $j = 2, l = 1$, and $i = 6$. Let $H_1 = H_{1,3}^{6\vee}$ and $H_2 = H_{1,4}^{6\vee}$ be vital hyperplanes. By construction the restriction of \mathcal{L}_6 to H_s is dominant, for $s = 1, 2$.

Claim 1. We may assume that $(\mathcal{L}_{6|H_s}, W_6)$ is free of fixed divisors for $s = 1, 2$.

Proof of the Claim. Consider $H_1 = H_{1,3}^{6\vee}$, $H_2 = H_{1,4}^{6\vee}$, and $H_3 = H_{1,5}^{6\vee}$. Then the restriction of \mathcal{L}_6 to H_s is dominant, for $s = 1, 2, 3$. Assume that for any pair of H_s , the linear system $\mathcal{L}_{6|H_s}$ has fixed divisors, say $F_s \subset H_s$. By construction $\text{Bs } \mathcal{L}_{6,Y_2}$ is a single point corresponding to the line $V_{1,2}^6$, hence $p_2 \notin F_s$. Then Lemma 3.6 yields

$$(1) \quad \text{Bs } \mathcal{L}_6 \supseteq V_{4,5}^6 \cup V_{3,5}^6 \cup V_{3,4}^6.$$

By Lemma 3.3 \mathcal{L}_6 is dominant at the first order at any Kapranov point and by equation (1) there are at least two points in $\text{Bs } \mathcal{L}_{6,Y_h} \cap E_h$, for $h = 3, 4, 5$. Let \mathcal{K}_h be the Kapranov set induced on E_h , then via Lemma 3.5 we conclude that

$$(2) \quad \text{Bs } \mathcal{L}_{6,Y_h} \supset \mathcal{K}_h \quad \text{for } h = 3, 4, 5.$$

Let \mathcal{M} be the movable part of the linear system $\mathcal{L}_{6|H_s}$. Let \mathcal{M}_{h,Y_h} be the strict transform of \mathcal{M} on the blow up Y_h . Then by Lemma 3.5 $\text{Bs } \mathcal{M}$ is of type Φ_1 , therefore $\mathcal{M}_{h,Y_h} \cap E_h = \emptyset$. This together with equation (2) yields that every vital line contained in H_s and passing through p_h is a fixed component of \mathcal{L}_6 . In particular we derive the contradiction

$$V_{2,h}^6 \subseteq \text{Bs } \mathcal{L}_6$$

□

If $\text{Bs } \mathcal{L}_6 \cap H_s$ is a single point p_h , then $\text{mult}_{p_h} \mathcal{L}_{6|H} = \deg \mathcal{L}_{6|H}$. Hence by Lemma 3.5, and 3.3 we know that

$$\text{mult}_{p_h} \mathcal{L}_6 = \text{mult}_{p_h} \mathcal{L}_{6|H} = \deg \mathcal{L}_{6|H} = \deg \mathcal{L}_6$$

so that

$$\text{mult}_{p_h} \mathcal{L}_6 = \deg \mathcal{L}_6$$

and we conclude by Proposition 3.1 that f factors via the forgetful map ϕ_h . Then every map, with connected fibers, from $\overline{M}_{0,5}$ is a forgetful map and we conclude that f itself is a forgetful map.

Assume that $\text{Bs } \mathcal{L}_6 \cap H_s$ is the full Kapranov set for $s = 1, 2$. Then consider the Cremona Transformation $\omega_5^{\mathcal{K}}$. Let $H'_s = \omega_5^{\mathcal{K}}(H_s)$, for $s = 1, 2$.

Claim 2. We may assume that the restricted linear system $\mathcal{L}_{5|H'_1}$ has not fixed divisors.

Proof of the Claim. Assume that $\mathcal{L}_{5|H'_s}$ has fixed divisors, for $s = 1, 2$. The linear system \mathcal{L}_6 is dominant at the first order in any Kapranov point, then the only possible fixed divisor of $\mathcal{L}_{5|H'_s}$ is $V_{2,5-s}^5$. This forces, as in the previous Claim,

$$\text{Bs } \mathcal{L}_{5,2} \supset \mathcal{K}_2,$$

where \mathcal{K}_2 is the Kapranov set induced on the exceptional divisor E_2 . Then as before we have

$$\text{Bs } \mathcal{L}_5 \supset V_{2,6}^5,$$

a contradiction. \square

The restriction $\omega_{5|H_s}^{\mathcal{K}}$ is a standard Cremona transformation of \mathbb{P}^2 . Hence the claim shows that $\text{mult}_{p_6} \mathcal{L}_{5|H'_1} = \deg \mathcal{L}_5$, and we conclude as above that f factors through a forgetful map. This concludes the $n = 6$ case. We are ready for the proof of the following:

Theorem 3.7. *Let $f : \overline{M}_{0,n} \rightarrow \mathbb{P}^1 \cong \overline{M}_{0,4}$ be a non constant morphism with connected fibers. Then f is a forgetful map.*

Proof. We prove the claim by induction on n . We already discussed the $n \leq 6$ case. We have to prove the result for $n \geq 7$. Let $f_n : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ be a Kapranov map with $\mathcal{K} = \{p_1, \dots, p_{n-1}\}$. Let $\mathcal{L}_n = (L_n, V_n) = f_n(f^*(\mathcal{O}(1)))$ be the associated linear system. Then the linear system \mathcal{L}_n is a pencil of hypersurfaces, without fixed components, say $\mathcal{L} = \{A_1, A_2\}$, and it is an $M_{\mathcal{K}}$ -linear system.

From Lemma 3.3 we know that there exists $p_1 \in \mathcal{K} \cap \text{Bs } \mathcal{L}_n$ such that \mathcal{L}_n is dominant at the first order of p_1 . From Lemma 3.3 we get:

$$(3) \quad m = \text{mult}_{p_1} A_1 = \text{mult}_{p_1} A_2 \text{ and } \text{mult}_{p_1} \mathcal{L} = \text{mult}_{p_1} \mathcal{L}_{|H_{h,k}^{n \vee}} \text{ for } h, k \neq 1.$$

Let $\epsilon_j : Y_j \rightarrow \mathbb{P}^{n-3}$ be the blow up of p_j with exceptional divisor E_j , for $j = 1, \dots, n-1$. By induction and Lemma 3.5, after possibly passing to another Kapranov map, we may assume that $\text{Bs } \mathcal{L}_{n,Y_1}$ is a codimension two linear space, and $\epsilon_1(\text{Bs } \mathcal{L}_{n,Y_1}) = V_{1,\dots,n-4}^n$.

From Lemma 3.4 we deduce that if H is a vital hyperplane containing p_1 but not containing $V_{1,\dots,n-4}^n$, the restriction of \mathcal{L}_n to H is dominant.

Claim 3. We may choose H in such a way that $\mathcal{L}_{n|H}$ is free of fixed divisors.

Proof of the Claim. Let $H_1 = H_{n-4,n-3}^{n \vee}$ and $H_2 = H_{n-5,n-2}^{n \vee}$ be two vital hyperplanes. Assume that the restriction of \mathcal{L}_n to H_i has a not empty fixed divisor F_i . Then from Lemma 3.6 the support of F_1 and of F_2 are respectively $V_{2,\dots,n-5,n-2,n-1}^n$ and $V_{2,\dots,n-6,n-4,n-3,n-1}^n$.

Lemma 3.3, applied to the point p_{n-1} , yields that \mathcal{L}_n dominates p_{n-1} at first order. Let \mathcal{K}_{n-1} be the induced Kapranov set on E_{n-1} . Then $\text{Bs } \mathcal{L}_{n,E_{n-1}} \cap E_{n-1}$ has two irreducible components meeting in codimension 4. On the other hand by Lemma 3.5 $\text{Bs } \mathcal{L}_{n,E_{n-1}} \cap E_{n-1}$ is of type Φ_1 , a contradiction. \square

Let H be a vital hyperplane such that the restriction of \mathcal{L}_n to H is dominant and $\mathcal{L}_{n|H}$ is free of fixed divisors. Then by Lemma 3.5 and Lemma 3.3 there is a Kapranov point $p_h \in H$ such that

$$\text{mult}_{p_h} \mathcal{L}_n = \text{mult}_{p_h} \mathcal{L}_{n|H} = \deg \mathcal{L}_{n|H} = \deg \mathcal{L}_i$$

so that

$$\text{mult}_{p_h} \mathcal{L}_i = \deg \mathcal{L}_i$$

We conclude by Proposition 3.1 that f factors via the forgetful map ϕ_h . That is $f = g \circ \phi_h$ for some morphism, with connected fibers, $g : \overline{M}_{0,n-1} \rightarrow \mathbb{P}^1$. By induction hypothesis g is a forgetful map. Henceforth f is forgetful. \square

As a further attempt to make the result and its main ideas clearer we add a proof of Theorem 3.7 kindly suggested by James McKernan and Jenia Tevelev, [McT]. We thank James and Jenia for their help in translating our projective arguments into a better known dictionary and also to produce a proof that shows at best the “almost toric” nature of $\overline{M}_{0,n}$.

Proof of Theorem 3.7 ([McT]). We proceed by induction on n . We may assume that $n \geq 7$. Let f_{ij} be the restriction of f to $\delta_{ij} := E_{i,j} \simeq \overline{M}_{0,n-1}$. There are two cases:

- (1) f_{ij} is never constant.
- (2) f_{ij} is constant, for at least one pair $\{i, j\}$.

Suppose we have (1). We will derive a contradiction. By induction, we know that each f_{ij} is a composition of a forgetful map $f'_{ij} : \delta_{ij} \rightarrow \mathbb{P}^1$ and a finite morphism $g_{ij} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Notice that forgetful maps f'_{ij} and f'_{kl} agree on intersections $\delta_{ij} \cap \delta_{kl}$ each time these divisors have a non-empty intersection, i.e. when $\{i, j\}$ and $\{k, l\}$ do not contain common elements. Indeed, both f'_{ij} and f'_{kl} restrict to some forgetful maps $\delta_{ij} \cap \delta_{kl} \simeq \overline{M}_{0,n-2} \rightarrow \overline{M}_{0,4} \simeq \mathbb{P}^1$. But

$$(f'_{ij})^* \mathcal{O}_{\mathbb{P}^1}(a) \simeq (f_{ij})^* \mathcal{O}_{\mathbb{P}^1}(1) \simeq (f_{kl})^* \mathcal{O}_{\mathbb{P}^1}(1) \simeq (f'_{kl})^* \mathcal{O}_{\mathbb{P}^1}(b)$$

for some positive integers a and b and a forgetful map $\overline{M}_{0,n-2} \rightarrow \overline{M}_{0,4} \simeq \mathbb{P}^1$ is uniquely determined by the pull-back of $\mathcal{O}_{\mathbb{P}^1}(1)$ (up to a multiple).

There are two cases, up to the obvious symmetries,

$$f'_{12} = \begin{cases} \pi_{3,4,5,6} \\ \pi_{\{1,2\},3,4,5} \end{cases}$$

Consider f_{67} . Up to even more symmetries, we must have

$$f'_{67} = \begin{cases} \pi_{3,4,5,\{6,7\}} & \text{if } f'_{12} = \pi_{3,4,5,6} \\ \pi_{2,3,4,5} & \text{if } f'_{12} = \pi_{\{1,2\},3,4,5} \end{cases}$$

Possibly switching $\{1, 2\}$ and $\{6, 7\}$ we might as well assume that

$$f'_{12} = \pi_{\{1,2\},3,4,5} \quad \text{and} \quad f'_{67} = \pi_{2,3,4,5}.$$

It follows that f restricted to both $\delta_{12} \cap \delta_{34}$ and $\delta_{34} \cap \delta_{67}$ is constant. But then $f'_{34} = \pi_{1267}$ and so $f'_{15} = \pi_{\{1,5\},2,6,7}$. On the other hand $f'_{15} = \pi_{\{1,5\},2,3,4}$, a contradiction.

So we must have (2). Assume that f contracts, say, δ_{1n} . Let $f_n : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ be the Kapranov map associated to the Kapranov set $\{p_1, \dots, p_{n-1}\}$. That is, the map that blows up $n-1$ points $\{p_1, \dots, p_{n-1}\}$ in linear general position, and every

linear space spanned by these points. Then f_n contracts $\delta_{1,n}$ to the point p_1 . Let $\psi: L_n \rightarrow \mathbb{P}^{n-3}$ be the birational morphism which blows up every linear space blown up by π , except those which contain p_1 . Notice that L_n is a toric variety, and there is a birational morphism $\varphi: \overline{M}_{0,n} \rightarrow L_n$ which factors $f_n = \psi \circ \varphi$. This yields an induced rational map $g = f \circ \varphi^{-1}: L_n \dashrightarrow \mathbb{P}^1$. Then the rational map $g: L_n \dashrightarrow \mathbb{P}^1$ is

- (a) regular at p_1 ;
- (b) has a base locus of codimension 2 (as for any rational pencil);
- (c) has a base locus contained in the indeterminacy locus of the birational map $\varphi^{-1}: L_n \dashrightarrow \overline{M}_{0,n}$. The latter is the union of linear subspaces passing through p_1 (in the Kapranov model).

It follows that the map $g: L_n \dashrightarrow \mathbb{P}^1$ is actually regular. As for any morphism, with connected fibers, to \mathbb{P}^1 , it is given by a complete linear series. Therefore it is a toric morphism. To conclude, we prove that it is one of the forgetful maps by studying the induced map of fans.

The fan F_n of L_n is obtained by taking the standard fan for \mathbb{P}^{n-3} (with rays R_1, \dots, R_{n-2} , of which the first $n-2$ correspond to coordinate hyperplanes) followed by its barycentric subdivision. A toric morphism $L_n \rightarrow \mathbb{P}^1$ corresponds to a linear map $g: \mathbb{R}^{n-3} \rightarrow \mathbb{R}$ that sends each cone of F_n to either $\{0\}$, the positive ray \mathbb{R}_+ , or the negative ray \mathbb{R}_- . We may assume without loss of generality that $g(R_1) = \mathbb{R}_+$ and $g(R_2) = \mathbb{R}_-$. The fan of L_n contains the ray $C = R_1 + R_2$ and we should have $g(C) = 0$. Therefore, g sends primitive generators of R_1 and R_2 to opposite vectors v_1 , and v_2 in \mathbb{R} . We claim that $g(R_i) = 0$ for $i > 2$. Assuming the claim, we then have $v_1, v_2 = \pm 1$ and the toric morphism is a resolution of the linear projection from the intersection of the first two coordinate hyperplanes, which corresponds to one of the forgetful maps.

Back to the claim, and arguing by contradiction, suppose that $g(R_3) = \mathbb{R}_+$. Then $g(-R_3) = \mathbb{R}_-$. But $-R_3$ is the barycenter of the top-dimensional cone of F_n spanned by all R_i for $i \neq 3$. Then F_n contains a cone with rays R_1 , and $-R_3$, which does not map to any cone. \square

Remark 3.8. *It is interesting to note that the space L_n is a moduli space in its own right, introduced by Losev and Manin [LM], and the morphism $\overline{M}_{0,n}$ to L_n has a natural modular interpretation.*

Corollary 3.9. *Let $f: \overline{M}_{0,n} \rightarrow \mathbb{P}^1 \cong \overline{M}_{0,4}$ be a non constant morphism. Then f factors through a forgetful map and a finite morphism.*

Proof. Taking the Stein factorization and the normalization, as in Lemma 3.5, we reduce the claim to Theorem 3.7. \square

4. MORPHISMS TO $\overline{M}_{0,r}$

In this final section we apply Theorem 3.7 to deduce that in fact any surjective morphism $f: \overline{M}_{0,n} \rightarrow \overline{M}_{0,r}$ is a forgetful map. As a corollary we compute the automorphisms group of $\overline{M}_{0,n}$.

Theorem 4.1. *Let $f: \overline{M}_{0,n} \rightarrow \overline{M}_{0,r}$ be a dominant morphism with connected fibers. Then f is a forgetful map.*

Proof. We prove the Theorem by induction on r . The first step of the induction process is the content of Theorem 3.7.

Let us fix a Kapranov map $f_r : \overline{M}_{0,r} \rightarrow \mathbb{P}^{r-3}$ and consider the forgetful map

$$\phi_{r-1} : \overline{M}_{0,r} \rightarrow \overline{M}_{0,r-1},$$

the Kapranov map $f_{r-1} : \overline{M}_{0,r-1} \rightarrow \mathbb{P}^{r-4}$ and the projection $\pi : \mathbb{P}^{r-3} \dashrightarrow \mathbb{P}^{r-4}$ given by the linear system $\Lambda_{r-1} = |\mathcal{O}_{\mathbb{P}^{r-3}}(1) \otimes \mathcal{I}_{p_{r-1}}|$. Then by induction hypothesis $(\phi_{r-1} \circ f) : \overline{M}_{0,n} \rightarrow \overline{M}_{0,r-1}$ is dominant and with connected fibers, hence a forgetful map. This means that we can choose a Kapranov map $f_n : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ such that

$$f_{n*}((f_r \circ f)_*^{-1}(\Lambda_{r-1})) \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|.$$

Recall that, from Proposition 3.1, we obtain the thesis if we show that

$$f_{n*}((f_r \circ f)^*(\mathcal{O}_{\mathbb{P}^{r-3}}(1))) \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|.$$

By construction we have $\Lambda_{r-1} = |\mathcal{O}_{\mathbb{P}^{r-3}}(1) \otimes \mathcal{I}_{p_{r-1}}|$ and $f_r^{-1}(p_{r-1}) = E_{r,r-1}$. To conclude it is enough to show that $f^*(E_{r,r-1})$ is f_n -exceptional.

The following diagram, will help us along the proof,

$$\begin{array}{ccccc} \overline{M}_{0,n} & \xrightarrow{f} & \overline{M}_{0,r} & \xrightarrow{\phi_{r-1}} & \overline{M}_{0,r-1} \\ \downarrow f_n & & \downarrow f_r & & \downarrow f_{r-1} \\ \mathbb{P}^{n-3} & \xrightarrow{\varphi} & \mathbb{P}^{r-3} & \xrightarrow{\pi} & \mathbb{P}^{r-4} \end{array}$$

Let \mathcal{L} be the linear system associated to the map

$$(\pi \circ \varphi) = (f_{r-1} \circ \phi_{r-1} \circ f \circ f_n^{-1}).$$

By induction hypothesis we may assume that f_n is such that $\mathcal{L} = |\mathcal{O}(1) \otimes \mathcal{I}_P|$, where $P = \langle p_{r-1}, \dots, p_{n-1} \rangle$. We fix notations in such a way that $(\pi \circ \varphi)(p_j) = p_j$, for $j < r-1$, and $\pi(p_j) = p_j$, for $j < r-1$.

For any $E_{j,r} \neq E_{(r-1),r}$ the map $\phi_{r-1|E_{j,r}} : \overline{M}_{0,r-1} \rightarrow \overline{M}_{0,r-1}$ is a forgetful map onto $\overline{M}_{0,r-2}$. Then for any $E_{i,r} \subset \overline{M}_{0,r}$, with $i < r-1$, we have that

$$f^*(E_{i,r}) = (\phi_{r-1} \circ f)^*(E_{i,r-1}) = E_{i,n}$$

This shows that $f^*(E_{i,r})$ is f_n -exceptional for $i < r-1$.

Notice that, once having fixed f_r and chosen the forgetful map ϕ_{r-1} , we have found f_n such that $f^*(E_{i,r}) = E_{i,n}$ for $i < r-1$.

We are assuming $r \geq 5$ hence, once we fix the Kapranov map $f_r : \overline{M}_{0,r} \rightarrow \mathbb{P}^{r-3}$ there are at least 4 possible forgetful maps $\phi_i : \overline{M}_{0,r} \rightarrow \overline{M}_{0,r-1}$, with $i < r$. To any such ϕ_i we may associate a Kapranov map $f_{n_i} : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ in such a way that $f^*(E_{j,r}) = E_{j,n_i}$, for $j \neq i$. On the other hand, the divisor $E_{i,j} \subset \overline{M}_{0,n}$ is sent to a point only by f_i and f_j . Then we may assume that $n_1 = n_2 = n$. The image of divisors $E_{i,r}$ via $f_{n*}f^*$ does not depend on the map ϕ_i . Therefore $f_{n*}f^*(E_{i,r})$ is a point for any $i = 1, \dots, r-1$.

By definition

$$f_r^*(\mathcal{O}(1)) = f_{r*}^{-1}\Lambda_{r-1} + E_{(r-1),r}$$

hence we have

$$\mathcal{L} = f_{n*}((f_r \circ f)^*(\mathcal{O}(1))) = f_{n*}((f_r \circ f)^*(\Lambda_{r-1})) \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|.$$

and φ is given by a linear system of hyperplanes. This is equivalent to our statement by Proposition 3.1 \square

This easily extends to morphisms onto products of \overline{M}_{0,r_i} .

Corollary 4.2. *Let $f : \overline{M}_{0,n} \rightarrow \overline{M}_{0,r_1} \times \dots \times \overline{M}_{0,r_h}$ be a dominant fiber type morphism with connected fibers. Then f is a forgetful map.*

Proof. It is enough to compose f with the projection onto the factors. \square

From Theorem 4.1 an automorphism of $\overline{M}_{0,n}$ must preserve all forgetful maps. This gives a very strong condition on the induced linear system of \mathbb{P}^{n-3} . We are ready to prove the main result on $\text{Aut}(\overline{M}_{0,n})$. This is classical for $n = 5$.

Theorem 4.3. *Assume that $n \geq 5$. Then $\text{Aut}(\overline{M}_{0,n}) = S_n$, the symmetric group on n elements.*

Proof. Let $g \in \text{Aut}(\overline{M}_{0,n})$ be an automorphism. Let $\phi_i : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-1}$ be the i -th forgetful map. Then by Theorem 4.1 $g \circ \phi_i$ is a map forgetting an index $j_i \in \{1, \dots, n\}$.

This means that we can associate to g the permutation $\{j_1, \dots, j_n\} \in S_n$. Let $\chi : \text{Aut}(\overline{M}_{0,n}) \rightarrow S_n$ be the associated map. The map χ is a surjective morphisms. A simple transposition is realized by the standard Cremona transformations we recalled in Definition 2.1.

The main point is to determine the kernel. Assume that $\chi(g) = 1$. That is $g \circ \phi_i$ is forgetting the i -th index for any $i \in \{1, \dots, n\}$. Fix a Kapranov map $f_n : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$. The automorphism g induces a Cremona transformation γ_n on \mathbb{P}^{n-3} that stabilizes the lines through the Kapranov points and also the rational normal curves through \mathcal{K} . Let $\mathcal{H}_n \subset |\mathcal{O}(d)|$ be the linear system associated to γ_n . Let $l_i \subset \mathbb{P}^{n-3}$ be a general line through p_i and Γ_n a general rational normal curve through \mathcal{K} . Then we have

$$\deg(\gamma_n(l_i)) = d - \text{mult}_{p_i} \mathcal{H}_n = 1,$$

for any $i \in \{1, \dots, n-1\}$, and

$$\deg(\gamma_n(\Gamma_n)) = (n-3)d - \sum_{i=1}^{n-1} \text{mult}_{p_i} \mathcal{H}_n = n-3.$$

These yield

$$n-3 = (n-3)d - (n-1)(d-1)$$

and finally $d = 1$. That is γ_n is a projectivity that fixes $n-1$ points. Then γ_n and henceforth g are the identity. \square

REFERENCES

- [Bo] M. Bolognesi *Forgetful linear systems on the projective space and rational normal curves over $\overline{M}_{0,2n}^{\text{GIT}}$* ArXiv:alg-geom/0909.0151.
- [BM] A. Bruno, M. Mella *Fiber type morphisms of $\overline{M}_{0,n}$* in preparation (2010).
- [CT] A. Castravet, J. Tevelev *Exceptional loci on $\overline{M}_{0,n}$ and hypergraph curves* ArXiv:alg-geom/0809.1699.
- [EK] C. J. Earle, I. Kra *On isometries between Teichmüller spaces* Duke Math. J. V.41, No. 3 (1974), 583-591.
- [FG] G. Farkas, A. Gibney *The Mori cones of moduli spaces of pointed curves of small genus*. Trans. Amer. Math. Soc. 355 (2003), 1183-1199.

- [Gi] A. Gibney, *The Fibrations of the Moduli Space of Stable n -Pointed Curves of Genus g , $M(\text{sub } g, n)$* , Phd thesis University of Texas at Austin (2000)
- [GKM] A. Gibney, S. Keel, I. Morrison *Towards the ample cone of $\overline{M}_{g,n}$* Journal of Amer. Math. Soc., 2 (2001), 273-294.
- [Ka] M. Kapranov *Veronese curves and Grothendieck-Knudsen moduli spaces $\overline{M}_{0,n}$* . Jour. Alg. Geom., 2 (1993), 239-262.
- [KM] S. Keel, J. M^cKernan *Contractible extremal rays of $\overline{M}_{0,n}$* . ArXiv:alg-geom9607009.
- [Ko] M. Korkmaz *Automorphisms of complexes of curves on punctured spheres and on punctured tori* Topology and its Applications Volume 95, Issue 2 (1999) 85-111.
- [La] P. Larsen *Fulton's conjecture for $\overline{M}_{0,7}$* . ArXiv:alg-geom0912310.
- [LM] A. Losev and Y. Manin, *New moduli spaces of pointed curves and pencils of flat connections*, Michigan Math. J. Volume 48, Issue 1 (2000), 443-472
- [M^cT] J. M^cKernan, J. Tevelev *private communication*
- [Mo] I. Morrison *Mori theory of moduli spaces of stable curves*. Projective Press, New York 2007.
- [Ro] H.L. Royden *Automorphisms and isometries of Teichmüller spaces* in Advances in the theory of Riemann surfaces Ed. by L. V. Ahlfors L. Bers H. M. Farkas R. C. Gunning I. Kra H. E. Rauch Annals of Math. Studies No.66 (1971), 369-383.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ ROMA TRE, L.GO S.L.MURIALDO, 1,
00146 ROMA ITALIA

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI FERRARA, VIA MACHIAVELLI 35,
44100 FERRARA ITALIA

E-mail address: `bruno@mat.uniroma3.it`

E-mail address: `mll@unife.it`